# Zeta Functions for Families of Calabi–Yau n-folds with Singularities

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**Abstract**

We consider families of Calabi–Yau $n$-folds containing singular fibres and study relations between the occurring singularity structure and the decomposition of the local Weil zeta-function. For 1-parameter families, this provides new insights into the combinatorial structure of the strong equivalence classes arising in the Candelas–de la Ossa–Rodrigues-Villegas approach for computing the zeta-function. This can also be extended to families with more parameters as is explored in several examples, where the singularity analysis provides correct predictions for the changes of degree in the decomposition of the zeta-function when passing to singular fibres. These observations provide first evidence in higher dimensions for Lauder's conjectured analogue of the Clemens–Schmid exact sequence.

## 1 Introduction

After a decade and a half of string theorists studying Calabi–Yau manifolds over fields of characteristic zero, particularly in the context of mirror symmetry, Candelas, de la Ossa and Rodrigues-Villegas [CdOV1] began the exploration of arithmetic mirror symmetry. Calabi–Yau manifolds over finite characteristic thus became objects of interest to physicists as well as mathematicians. After the discovery that the moduli spaces of all known Calabi–Yau manifolds form a web linked via conifold transitions [GH], the interest on the part of physicists decreased significantly concerning more complicated singularities which occur at other interesting points in the complex structure moduli space. However, newer results such as [KLS] suggest that it might be worthwhile to reconsider this and ask questions such as: Is string theory viable on spaces with singularities with high Milnor numbers and even non-isolated singularities? Can the D-brane interpretation of conifold (i.e. ordinary double points) transitions by Greene, Strominger and Morrison [S, GMS] be extended to what would be more complicated phase transitions? Questions of this type have not been considered very deeply yet - in part, because the study of singularities with more structure requires different methods. In this article, we

want to start an approach in this direction by specifically studying properties at the singular fibres of families of Calabi–Yau varieties. In [KLS] the first question was addressed by finding points in the moduli space where the singular Calabi–Yau manifolds exhibited modularity, (i.e. are their cohomological $L$-series completely determined by certain modular cusp forms) as a consequence of the rank of certain motives decreasing in size at singularities. For an overview of Calabi–Yau modularity the reader may consult [HKS]. Our approach, which enables the specification of exactly how much the degree of the contribution to the zeta function associated to each strong orbit (which in turn is directly related to motive rank) decreases, would further aid such investigations.

The local Weil zeta-function for certain families of Calabi–Yau varieties of various dimensions decomposes into pieces parametrized by monomials which are related to the toric data of the Calabi–Yau varieties [CdOV1, CdOV2, CdO, K04, K06]. It was shown in these papers that this decomposition points to deeper structures, since these monomials can also be related to the periods which satisfy Picard-Fuchs equations. Away from the singular fibres, this phenomenon of a link to $p$-adic periods was explained for one-parameter families using Monsky–Washnitzer cohomology in [Kl]. The families considered there all have the property that one distinguished member of each family is a diagonal variety of Fermat type; these are very accessible to explicit computations and are known to possess decompositions in terms of Fermat motives [GY95, KY].

At certain values of the parameter, the corresponding variety becomes singular, and it was observed in [K04, K06] that the degree of the contribution to each piece decreases according to the types of singularities encountered in explicit examples. In order to test, whether the observations in [K04, K06] concerning the degenerations of the zeta functions for singular Calabi–Yau varieties hold more generally, we analyse the discriminant locus and singularity structure for general 1-parameter and some explicit 2-parameter families of Calabi–Yau varieties with distinguished fibre of Fermat-type and compare the results to the structure of their zeta functions. In particular, this provides strong evidence for conjectures connecting the numbers and types of singularities in the discriminant locus with certain combinatorial arguments arising in motivic and zeta function considerations and proves the facts for the considered cases by a direct comparison. In all cases with isolated singularities the total Milnor number of the singularities is given precisely by the degeneration in the degree of the various parts of the zeta function. Observations on finer combinatorial properties of the decomposition are also possible; for the 1-parameter families, the decomposition of the singular locus and the Milnor numbers of the types of singularities occuring are reflected in the analysis of the structure of this degeneration. For these considerations, the choice of using Dwork's original approach for computing the zeta-function was influenced by two constraints: by the presence of isolated singularities in the cases of interest and by the goal to also study higher-dimensional examples, which basically rules out explicit resolution of singularities in many cases due to the intrinsic complexity of the algorithm.

After fixing notation and stating references for standard facts about the local Weil zeta function at good primes in Section 2, we first analyze the occurring singularities in detail in Section 3. There we focus on combinatorial aspects in the calculations, which by themselves do not seem very exciting at first glance, but reoccur from a different perspective in the computation of the zeta functions for the corresponding singular fibres in the subsequent section. This correspondence is then explored further in Section 5 for explicit examples of 2-parameter families and leads to the conjectures at the end of the article linking the singularity structure and the decomposition of the zeta function. If these conjectures hold, then a singularity analysis in the singular fibres coupled with a calculation of the zeta function away from the singular fibres already provides a large amount of vital information on the zeta function at the singularities by using well established standard methods of singularity theory and of point counting.

The authors would like to thank the members of the Institut für Algebraische Geometrie and the Graduiertenkolleg 'Analysis, Geometrie und Stringtheorie' for the good working atmosphere and the insightful discussions. All computations of the singularity analysis were done in Singular [DGPS], for the zeta-function calculations Mathematica [Mth] was used.

# 2 Facts about the Weil zeta function

A pair of reflexive polyhedra $(\Delta, \Delta^*)$ is known to give rise to a pair of mirror Calabi–Yau families $(\hat{V}_{f,\Delta}, \hat{V}_{f,\Delta^*})$. In this setting, Batyrev proved that topological invariants such as the Hodge numbers could be written in terms of the toric combinatorial data given by the reflexive polytopes. For the case of families of Calabi–Yau varieties which are deformations of a Fermat variety, the data of the reflexive polytope is encoded in certain monomials. For a detailed treatment of toric constructions of mirror symmetric Calabi–Yau manifolds see [Bat, CK].

First we recall a few standard definitions: the arithmetic structure of Calabi–Yau varieties can be encoded in the congruent or local zeta function. The Weil Conjectures (proven by Deligne [Del2] in 1974) show that the local zeta function is a rational function determined by the cohomology of the variety.

**Definition 1 (Local zeta function)** *The local zeta function for a smooth projective variety $X$ defined over $\mathbb{F}_p$ is defined as follows:*

$$\zeta(X/\mathbb{F}_p, t) := \exp\left(\sum_{r\in\mathbb{N}} \#(X/\mathbb{F}_{p^r})\frac{t^r}{r}\right), \qquad (1)$$

*where $\#(X/\mathbb{F}_{p^r})$ is the number of rational points of the variety.*

For families of Calabi–Yau manifolds in weighted projective space the local zeta function can be computed in various ways, we however shall utilise exclusively, methods first developed by Dwork in his proof of the

rationality part of the Weil conjectures [Dw1, Dw2]. We thus use Gauss sums composed of the additive Dwork character, $\Theta$ and the multiplicative Teichmüller character, $\omega^n(x)$:

$$G_n = \sum_{x \in \mathbb{F}_p^*} \Theta(x)\omega^n(x). \tag{2}$$

When a variety is defined as the vanishing locus of a polynomial $P \in k[X_1, \ldots, X_n]$, where $k$ is a field, a non-trivial additive character like Dwork's character can be exploited to count points over $k$. Since $\Theta(x)$ is a character:

$$\sum_{y \in k} \Theta(yP(x)) = \begin{cases} 0 & \text{if } P(x) \neq 0, \\ q := \mathrm{Card}(k) & \text{if } P(x) = 0 \ ; \end{cases} \tag{3}$$

hence

$$\sum_{x_i \in k} \sum_{y \in k} \Theta(yP(x)) = q\#(X/\mathbb{F}_{p^r}) \ , \tag{4}$$

The above equation can be expressed in terms of Gauss sums which are amenable to computation via the Gross-Koblitz formula [GK]. All zeta function computations in this paper use an implementation of this method on Mathematica developed in [K04, K06]. In our context, the choice of this method was mainly influenced by the fact that it is also suitable for treating singular Calabi–Yau varieties, whereas most other approaches are restricted to the non-singular case. Lauder's extension of the deformation method [L2] to the singular case relies on the existence of an analogue of the Clemens-Schmid exact sequence in positive characteristic which is currently only conjectural.

The decomposition of the number of points and hence the zeta function into parts labelled by strong $\beta$-classes, $\mathcal{C}_\beta$, was shown in [CdOV1, CdOV2, K04, K06] as a direct consequence of these methods [1].

$$\zeta(t,a) = \zeta_{\mathbf{c}onst}(t) \prod_{\mathcal{C}_\beta} \zeta_{\mathcal{C}_\beta}(t,a)$$

where $\zeta_{\mathbf{c}onst}(t)$ is a simple term, which is independent of the parameter $a$, and the $\beta$-classes are defined as follows:

**Definition 2 (Strong motivic $\beta$-equivalence classes)** *For a given set of weights, $\mathbf{w} = (w_1, \ldots, w_n)$, $d = \sum_i w_i$, $w_i | d \ \forall i$, identify the set of all monomials with the set of all exponents of the monomials. We now consider a subset thereof defined as*

$$\mathcal{M} := \mathcal{M}(\mathbf{w}) := \left\{ \mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n) \in \prod_{i=1}^{n} w_i\mathbb{Z}/d\mathbb{Z} \ \mid \mathbf{x} \cdot \mathbf{w} = ld, \ l \in \mathbb{Z} \right\}.$$

*It is easy to see that $0 \leq l \leq n-1$.*
*Let $l(\mathbf{x}) := \mathbf{x} \cdot \mathbf{w}/d$. Given a $\beta \in \mathcal{M}$ with $l(\beta) = 1$, we can quotient out the set $\mathcal{M}$ with the equivalence relation $\sim_\beta$ on monomials, where*

$$\mathbf{x} \sim_\beta \mathbf{y} \Leftrightarrow \mathbf{y} = \mathbf{x} + t\beta, \ t \in \mathbb{Z},$$

[1] These papers do not explicitly refer to'strong equivalence classes', the term was coined later by Kloosterman in [Kl]

*From now on we shall assume (unless otherwise stated) that the ith exponent of each monomial is taken* $\mod \frac{d}{w_i}$. *The equivalence classes,* $\mathcal{C}_\beta$, *thus obtained shall be referred to as the* **strong** $\beta$ **-equivalence classes**.

**Remark 3** *For families of Calabi–Yau varieties which are deformations of smooth varieties of Fermat type the toric data is equivalent to specifying the monomials* $\mathbf{x} \in \mathcal{M}$ *for which* $l(\mathbf{x}) = 1$*, see [CK].*

**Proposition 4 ([Kl])** *Considering smooth fibres of a* 1*-parameter family of Calabi–Yau varieties, the factor of the local zeta function associated to a* $\beta$*-class and the parameter value a is at worst a fractional power*

$$\zeta_{\mathcal{C}_\beta}(t,a) = \left(\frac{P(t,a)}{Q(t,a)}\right)^{\frac{r}{s}},$$

*where* $P(t,a)$ *and* $Q(t,a)$ *are polynomials and* $r, s \in \mathbf{Z}$*. We define*

$$deg(\zeta_{\mathcal{C}_\beta}(t,a)) := (deg(P) - deg(Q))\frac{a}{b}.$$

*This degree of the factor of the zeta function associated to each strong* $\beta$*-class, can be computed as the number of monomials in the class which do not contain* $\left(\frac{d}{w_i} - 1\right)$ *in its ith component.*

Kloosterman's explanation of the above-stated relation using Monsky-Washnitzer cohomology breaks down when the variety in question is singular. A key aim of this article is to explore the degenerations of the various pieces of the zeta function for singular fibres. More sophisticated theoretical tools such as limiting mixed Frobenius structures in rigid cohomology will be needed to explain the degenerations. Lauder [L2] provides a preliminary exploration of this through the introduction of a conjectured analogue of the Clemens-Schmid exact sequence, but his testing ground for the conjecture mostly consists of families of curves.

In this article we are able to supplement Lauder's examples through looking at singularities of higher-dimensional varieties, not just curves, as curves are prone to oversimplification due to their low dimensionality and could thus be misleading. All our results for 1-parameter families are applicable in all dimensions. Moreover, all arguments are explicit and no step requires desingularization, which would effectively have blocked the simultaneous view in all dimensions. In this article we intentionally only provide phenomenological (and for 2-parameter families also experimental) data, but no theoretical explanation for the observed correspondences, because we see it merely as the first step in this direction. We wish to disseminate the observations as soon as possible and would prefer to devote another article to the theoretical side in due time.

# 3 Singularity analysis for some families of Fermat-type Calabi–Yau n-folds

In this section, we collect data about the discriminant and the singularities of the fibres. To this end, we first consider general 1-parameter families in

detail and then proceed to general observations on 2-parameter families which establish the background for the explicit examples in Section 5.

### 3.1 1-parameter families

For the 1-parameter families, we can explicitly specify a Gröbner Bases for the relative Tjurina ideal w.r.t. a lexicographical ordering, where the parameter $a$ of the family is considered smaller than any of the variables. As a consequence, we can specify the discriminant of the family, count the number of singularities in each fibre over the base space and determine the Milnor numbers of the occurring singularities. A priori this is not very interesting, but later on it will turn out that the same kind of combinatorial data which arise here also appear in the computation of the Weil zeta function at singular fibres of the family. Moreover, we shall consider 2-parameter families later on, which specialize to such 1-parameter families, if one parameter is set to zero. For these considerations, we shall make use of the explicit calculations of this subsection.

Before stating the result explicitly, we need to recall one small observation which will yield a key argument in the proof:

**Lemma 5** *Consider a polynomial ring $R[x]$ over some (noetherian commutative) ring $R$ (with unit). Let $f = Ax^\alpha - C$, $g = Bx^\beta - D$ for some $A, B, C, D \in R$. Then the ideal $\langle f, g\rangle$ contains polynomials which we can symbolically write as*

$$\begin{array}{rcl} A^r D^s x^{\gcd(\alpha,\beta)} & - & C^r B^s, \\ C^{\frac{\beta}{\gcd(\alpha,\beta)}-r} B^{\frac{\alpha}{\gcd(\alpha,\beta)}-s} x^{\gcd(\alpha,\beta)} & - & A^{\frac{\beta}{\gcd(\alpha,\beta)}-r} D^{\frac{\alpha}{\gcd(\alpha,\beta)}-s} \\ A^{\frac{\beta}{\gcd(\alpha,\beta)}} D^{\frac{\alpha}{\gcd(\alpha,\beta)}} & - & C^{\frac{\beta}{\gcd(\alpha,\beta)}} B^{\frac{\alpha}{\gcd(\alpha,\beta)}} \end{array}$$

*where $r, s$ are integers arising from the Bézout identity $r\alpha - s\beta = \gcd(\alpha, \beta)$; to avoid ambiguities, we choose precisely the ones arising from the extended Euclidean algorithm as either $r$ and $s$ or as $\frac{\beta}{\gcd(\alpha,\beta)} - r$ and $\frac{\alpha}{\gcd(\alpha,\beta)} - s$ making sure that $r$ and $s$ are both positive integers.*

Using this, we can now state the main lemma of this section:

**Lemma 6** *Let $\mathcal{X} \subset \mathbb{P}_{w_1,\ldots,w_n}$, be the 1-parameter family of Calabi–Yau varieties*[2] *given by the polynomial*

$$F = \left(\sum_{i=1}^n x_i^{\frac{d}{w_i}}\right) + a \cdot \prod_{i=1}^k x_i^{\beta_i}$$

*where $d = \sum_{i=1}^n w_i = \sum_{i=1}^k \beta_i w_i$, $\beta_i \neq 0 \forall 1 \le i \le k$, and $\gcd(w_1, \ldots, w_n) = 1$. Let $\gamma := \gcd(\beta_1 w_1, \ldots, \beta_k w_k)$. Then the discriminant of the family is*

$$V\left(a^{\frac{d}{\gamma}} + (-1)^{\frac{d}{\gamma}-1} \frac{d^{\frac{d}{\gamma}}}{\prod_{i=1}^k \beta_i w_i^{\frac{\beta_i w_i}{\gamma}}}\right) \subset \mathbb{A}^1_{\mathbb{C}}.$$

[2] Note that up to permutation of variables any 1-parameter family of Calabi–Yau varieties with given zero fibre of Fermat type and perturbation term of weighted degree $d$ can be written in this form.

*In the respective fibre above each of the $\frac{d}{\gamma}$ points of the discriminant there are precisely*

$$\frac{\gcd(w_1,\dots,w_k)}{\prod_{i=1}^k w_i}\cdot d^{k-2}\cdot\gamma$$

*singularities with local equation*

$$\sum_{i=2}^k x_i^2+\sum_{i=k+1}^n x_i^{\frac{d}{w_i}}$$

*of Milnor number $\prod_{i=k+1}^n(\frac{d}{w_i}-1)$ and no further singularities.*

**Proof:**

Preparations:

As we are considering hypersurfaces here, the relative $T^1$ is of the form $(\mathbb{C}[a])[\underline{x}]/J$, where

$$J=\left\langle F,\frac{\partial F}{\partial x_1},\dots\frac{\partial F}{\partial x_n}\right\rangle$$

is the relative Tjurina ideal (Due to the weighted homogeneity and the resulting Euler relation, we can drop one of the $n+1$ generators.). More precisely, this ideal actually describes the relative $T^1$ of the affine cone over our family and we therefore need to ignore all contributions for which the associated prime is the irrelevant ideal. This is not difficult here, since intersection with any of the $k$ first coordinate hyperplanes immediately leads to an $\langle x_1,\dots,x_n\rangle$ primary ideal, and hence passage to any of the first $k$ affine charts immediately removes precisly the unwanted part, but nothing else. As we are in weighted projective space and want to count singularities, our choice of the appropriate affine charts needs a little bit of extra caution: a priori we count points before the identification and thus might obtain a multiple of the correct number. Hence the calculated number needs to be divided by the weight of the respective variable. To simplify the presentation of the subsequent steps, we choose the chart $x_1\neq 0$.

Gröbner Basis:

Our next step is to compute a Gröbner basis of the relative Tjurina ideal in this chart where $\alpha_i$ denotes $\frac{d}{w_i}$ to shorten notation. For the structure of the final result, it turns out to be most suitable to choose a lexicographical ordering with $x_2>\dots>x_n>a$.

$$\begin{aligned}
f_0 &= \left(\sum_{j=2}^n x_j^{\alpha_j}\right)+1+a\prod_{j=2}^k x_j^{\beta_j}\\
f_i &= \alpha_i x_i^{\alpha_i-1}+a\beta_i x^{\beta_i-1}\prod_{\substack{j=1\\ j\neq i}}^k x_j^{\beta_j}\quad\text{for}\quad 2\leq i\leq k\\
f_i &= \alpha_i x_i^{\alpha_i-1}\quad\text{for}\quad k+1\leq i\leq n
\end{aligned}$$

As $f_0-\sum_{i=2}^n\frac{1}{\alpha_i}x_if_i=a\frac{1}{\alpha_1}\left(\prod_{i=2}^k x_i^{\beta_i}\right)+1$, we may safely set

$$h_0=\frac{1}{\alpha_1}\left(a\prod_{i=2}^k x_i^{\beta_i}\right)+1$$

instead of the original $f_0$.
Forming $f_2x_2-\beta_2\alpha_1 h_0$ and the s-polynomials of the pairs $(f_2,h_0),\ldots,(f_k,h_0)$, we obtain new polynomials

$$h_i = x_i^{\alpha_i} - \frac{\beta_i\alpha_1}{\alpha_i} \quad 2 \leq i \leq k.$$

The leading monomials of these $h_i$, $2 \leq i \leq k$ and of the $f_i$, $k < i \leq n$, are obviously pure powers in the respective variables $x_i$. We shall use them later on when computing the discriminant.
Considering $h_0$ and $h_2$, we now apply Remark 5 (polynomial 1 or 2 respectively) and obtain a polynomial

$$g_2 = x_2^{\gcd(\beta_2,\alpha_2)} - \underbrace{\left(\frac{\beta_i\alpha_1}{\alpha_i}\right)}_{:=c_1}^{r} \cdot \left(a\prod_{i=3}^{k} x_i^{\beta_i}\right)^{s}$$

for suitable exponents $r, s \in \mathbb{N}$ as specified in the remark. Please note that the exponent of $x_2$, $\gcd(\beta_2,\alpha_2)$ can be written as $\frac{1}{w_2}\gcd(d,\beta_2 w_2)$. By polynomial 3 of the same remark

$$h_{0,new} = c_1^{\frac{\beta_2}{\gcd(\alpha_2,\beta_2)}} \cdot \left(\frac{1}{\alpha_1} a\prod_{i=3}^{k} x_i^{\beta_i}\right)^{\frac{\alpha_2}{\gcd(\alpha_2,\beta_2)}} - 1$$

In this expression, the use of properties of gcd shows that the exponent of $x_3$ is of the form $\frac{d}{\gcd(d,\beta_2 w_2)}$. Reducing all of the $h_i$ by $g_2$, we obtain polynomials which no longer depend on $x_2$, because all occurrences of $x_2$ in the $g_2$ were of the form $x_2^{\beta_2}$. We are hence in the situation to apply Remark 5 again, this time to $x_3$ and can eventually iterate the process $k-2$ times. This leads to polynomials of the form

$$g_i = x_i^{\frac{d}{w_i}\frac{\gcd(d,\beta_2 w_2,\ldots,\beta_i w_i)}{\gcd(d,\beta_2 w_2,\ldots,\beta_{i-1} w_{i-1})}} - c_i \cdot p_i(x_{i+1},\ldots,x_k)$$

for each $3 \leq i \leq k$.
To determine the discriminant we could now continue one step further, eliminating $x_k$, but here it is easier to observe (e.g. by explicit polynomial division) that for any polynomial $1-p(\underline{x},a)$, also every polynomial $1-p(\underline{x},a)^k$ is in the ideal. Applying this to $h_0$ and the $\frac{d}{\gamma}$-th power, where

$$\gamma = \gcd(\beta_1 w_1,\ldots,\beta_k w_k) = \gcd(d,\beta_2 w_2,\ldots,\beta_k w_k),$$

we obtain

$$h_{k+1} = 1 - \left(\frac{1}{\alpha_1} a\prod_{i=2}^{k} x_i^{\beta_i}\right)^{\frac{d}{\gamma}}.$$

But the exponents $\alpha_i$ of the leading monomials of the $h_i$ all divide $\beta_i\gamma$ for $2 \leq i \leq k$ by construction which allows reduction of $h_{k+1}$ by these and leads to the claimed expression

$$g_{n+1} = a^{\frac{d}{\gamma}} \frac{\prod_{i=1}^{k}(\beta_i w_i)^{\frac{\beta_i w_o}{\gamma}}}{d^{\frac{d}{\gamma}}} + (-1)^{\frac{d}{\gamma}-1}.$$

To finish the Gröbner basis calculation, let us first consider the set of polynomials $S = \{h_2, \ldots, h_n, g_2, \ldots, g_k, g_{n+1}\}$. For $2 \leq i \leq k$ we drop $h_i$ from it, if the $x_i$-degree of $g_i$ is strictly smaller than the one of $h_i$, otherwise we drop $g_i$. The resulting set then contains $n$ polynomials of which each of the first $n-1$ has a pure power of the respective variable $x_i$ as leading monomial, and the last element $g_{k+1}$ which has a leading monomial not involving any of the $x_i$. Hence this set obviously forms a Gröbner basis of some ideal, because all s-polynomials vanish by the product criterion. It then remains to show that the original polynomials $f_0, \ldots, f_n$ reduce to zero w.r.t. this set which can be checked by a straight forward but lengthy calculation.

Reading off the data:
It is clear that $a$ takes precisely the $\frac{d}{\gamma}$ values

$$\sqrt[\frac{d}{\gamma}]{\frac{d^{\frac{d}{\gamma}}}{\prod_{i=1}^{k}(\beta_i w_i)^{\frac{\beta_i w_i}{\gamma}}}} \cdot \zeta$$

where $\zeta$ runs through all the $\frac{d}{\gamma}$-th roots of unity. At each of these points in the base, we can obtain the number of singularities by plugging in the value for $a$ into $g_k$ and counting solutions, followed by the values for $a$ and $x_k$ into $g_{k-1}$ and so on, where $x_{k+1} = \cdots = x_n = 0$. This leads to the expression

$$\frac{1}{w_2}\gcd(d, \beta_2 w_2)\frac{d}{w_3}\frac{\gcd(d, \beta_2 w_2, \beta_3 w_3)}{\gcd(d, \beta_2 w_2)} \cdots \frac{d}{w_k}\frac{\gcd(d, \beta_2 w_2, \ldots, \beta_k w_k)}{\gcd(d, \beta_2 w_2, \ldots, \beta_{k-1} w_{k-1})}$$

for the number of singular points, which after simplification of the expression and multiplication by $\frac{\gcd(w_1, \ldots, w_k)}{w_1}$ (to take account of the identification of points in weighted projective space) leads to the claimed number. The multiplicity of each of these points is then given by the product of the powers of the variables $x_i$ in the polynomials $h_i$, $k+1 \leq i \leq n$. As the Gröbner basis generates the global Tjurina ideal of the fibre for each fixed value of $a$, the corresponding support describes the singular locus and the local multiplicity at each of the finitely many points is precisely the Tjurina number. By considering the corresponding local equations, we can then check that the Tjurina number and Milnor number coincide for the arising singularities.

**q.e.d.**

Considering the extreme cases of the families with the highest and lowest numbers of singularities, we obtain: plural

**Corollary 7** *Let $\mathcal{X} \subset \mathbb{P}_{w_1, \ldots, w_n}$ be the 1-parameter family of Calabi–Yau varieties given by the polynomial*

$$F = \left(\sum_{i=1}^{n} x_i^{\frac{d}{w_i}}\right) + a \cdot \prod_{i=1}^{n} x_i$$

*where* $d = \sum_{i=1}^{n} w_i$ *and* $\gcd(w_1, \ldots, w_n) = 1$*. Then the discriminant of the family is*

$$V\left(a^d + (-1)^{d-1}\frac{d^d}{\prod_{i=1}^{n} w_i^{w_i}}\right) \subset \mathbb{A}^1_{\mathbb{C}}.$$

*In the respective fibre, above each of the d points of the discriminant, there are precisely*

$$\frac{d^{n-2}}{\prod_{i=1}^{n} w_i}$$

*ordinary double points (with Milnor number* $\mu = 1$ *and Tjurina number* $\tau = 1$*) and no further singularities.*

**Corollary 8** *Let* $\mathcal{X} \subset \mathbb{P}_{w_1,\ldots,w_n}$ *be the 1-parameter family of Calabi–Yau varieties given by the polynomial*

$$F = \left(\sum_{i=1}^{n} x_i^{\frac{d}{w_i}}\right) + a x_1^{\frac{d-w_n}{w_1}} x_n$$

*where* $d = \sum_{i=1}^{5} w_i$ *and* $w_1 | w_n$*. Then the discriminant of the family is*

$$V\left(a^{\frac{d}{w_n}} + (-1)^{\frac{d}{w_n}-1}\frac{d^{\frac{d}{w_n}}}{w_n(d-w_n)^{\frac{d}{w_n}-1}}\right) \subset \mathbb{A}^1_{\mathbb{C}}.$$

*In the respective fibre above each of the* $\frac{d}{w_n}$ *points of the discriminant there is precisely* 1 *isolated singularity of which the local normal form (after moving to the coordinate origin) is*

$$\left(\sum_{i=2}^{n-1} x_i^{\frac{d}{w_i}}\right) + x_n^2$$

*with Milnor number* $\mu = \prod_{i=2}^{n-1}\left(\frac{d}{w_i} - 1\right)$.

## 3.2 Some particular 2-parameter families

In this case, the Gröbner basis of the relative Tjurina ideal is far too complicated to write down in general. Nevertheless, it is possible to follow the lines of some of the calculations of the previous subsection to specify and study the discriminant of some families. By analysis of the discriminant it is then possible to precisely classify the arising singularities in explicit families.

**Lemma 9** *Let* $\mathcal{X} \subset \mathbb{P}_{w_1,\ldots,w_n}$ *be the 2-parameter family of Calabi–Yau (n-2)-folds given by the polynomial*

$$F = \sum_{i=1}^{n} x_i^{\frac{d}{w_i}} + a\prod_{i=1}^{n} x_i + b x_1^{\beta_1} x_2^{\beta_2}$$

*where* $d = \sum_{i=1}^{n} w_i$ *and* $\beta_1 w_1 + \beta_2 w_2 = d$*. Then the discriminant of this 2-parameter family is reducible and its irreducible components can be sorted into two different kinds:*

- *Lines $L_i$ parallel to the a-axis, which are determined by the discriminant of*

$$x^{\frac{d}{w_2}} + bx^{\beta_2} + 1$$

- *A (possibly reducible) curve $C$ which can be specified as the resultant of*

$$a^d x_2^d - \frac{d^{d-2}}{\prod_{i=3}^n w_i^{w_i}} \left( \beta_1 b x_2 + \frac{d}{w_1} \right)$$

*and*

$$\frac{d}{w_2} x_2^{\frac{d}{w_2}} + (\beta_2 - \beta_1) b x_2^{\beta_2} - \frac{d}{w_1}.$$

**Proof:** As before, we choose a suitable affine chart, say $x_1 \neq 0$, and fix a lexicographical monomial ordering $x_n > \cdots > x_2 > a > b$. But here an explicit computation of a Gröbner basis of the Tjurina ideal cannot be performed in all generality. Instead, we can proceed analogous to the steps of the proof of Lemma 6 and obtain the following elements of the ideal:

$$\begin{aligned}
h_i &= \frac{d}{w_i} x_i^{\frac{d}{w_i}} - \beta_1 b x_2^{\beta_2} - \frac{d}{w_1} \forall 3 \leq i \leq n \\
h_2 &= \frac{d}{w_2} x_2^{\frac{d}{w_2}} + (\beta_2 - \beta_1) b x_2^{\beta_2} - \frac{d}{w_1} \\
h_0 &= a \prod_{i=2}^n x_i + \left( \beta_1 b x_2^{\beta_2} + \frac{d}{w_1} \right)
\end{aligned}$$

As before, we can again conclude that also

$$\left( a \prod_{i=2}^n x_i \right)^d - \left( \beta_1 b x_2^{\beta_2} + \frac{d}{w_1} \right)^d$$

is in the ideal and forming a normal form w.r.t. $h_3, \ldots, h_n$ then yields

$$\left( \beta_1 b x_2^{\beta_2} + \frac{d}{w_1} \right)^{\sum_{i=3}^n w_i} \cdot \left( a^d x_2^d \prod_{i=3}^n w_i^{w_i} - d^{d-2} \left( \beta_1 b x_2^{\beta_2} + \frac{d}{w_1} \right)^{w_1 + w_2} \right)$$

At this point, we can branch our computation and consider each factor separately.

$\underline{g_1 = \left( \beta_1 b x_2^{\beta_2} + \frac{d}{w_1} \right)}$: Here we directly obtain

$$g_2 = g_1 + h_2 = \frac{d}{w_2} x_2^{\frac{d}{w_2}} + \beta_2 b x_2^{\beta_2}$$

and

$$g_3 = \frac{w_1}{d} g_1 + \frac{w_2}{d} g_2 = x_2^{\frac{d}{w_2}} + b x_2^{\beta_2} + 1.$$

Therefore the resultant of $g_2$ and $g_3$ is also contained in the ideal. On the other hand, $g_2 = x_2 \frac{\partial g_3}{\partial x_2}$ and hence the above resultant is just the

discriminant of $g_3$ by the rules for computing resultants and the fact that $Res_{x_2}(g_3, x_2) = 1$.

$g_4 = a^d x_2^d \prod_{i=3}^n w_i^{w_i} - d^{d-2}\left(\beta_1 b x_2^{\beta_2} + \frac{d}{w_1}\right)^{w_1+w_2}$: As $g_4$ and $h_2$ are both in the ideal so is their resultant w.r.t. $x_2$ which describes the desired curve.

**q.e.d.**

On the basis of this lemma, it is now easy to treat interesting special cases, which we want to consider in a later section of this article, by a straight-forward computation. In order to treat such examples by the combinatorial algorithm for determining the zeta-function, the two perturbation monomials need to be in the same strong $\beta$-orbit in the sense that the orbit structure w.r.t. the second monomial refines the one w.r.t. the first monomial. As this is a rather restrictive condition on the possible choices of monomials, we only state a choice of three explicit examples in Section 5.

# 4 The influence of singularity data on the zeta function

In the previous section, we analysed the singularity structure of some 1- and 2-parameter families of Calabi–Yau varieties and, in particular, the structure of the Milnor algebra which encodes cohomological information about the singularities. Now we shift our focus to the computation of the local zeta-function for these families and re-encounter combinatorial data which we already saw in the previous section.

**Remark 10** *Recalling Definition 2 of strong motivic $\beta$-classes in $\mathcal{M}$, it is easy to show that each strong $\beta$-class, $\mathcal{C}_\beta$, is a set with cardinality $d_\beta$, where*

$$d_\beta = \underset{\beta_i \neq 0}{\mathrm{lcm}}(\mathrm{ord}(\beta_i)) = \underset{\beta_i \neq 0}{\mathrm{lcm}}\left(\frac{d}{\gcd(\beta_i w_i, d)}\right) = \frac{d}{\gcd_{\beta_i \neq 0}(\beta_i w_i)}.$$

*Hence the total number of strong $\beta$-classes, $\mathcal{O}_\beta$ is*

$$\mathcal{O}_\beta = |\mathcal{M}| \frac{\gcd_{\beta_i \neq 0}(\beta_i w_i)}{d}.$$

**Lemma 11** *Let $w_1, \ldots, w_n$ be a set of weights satisfying the conditions of Defintion 2. The total number of elements in $\mathcal{M}$ is*

$$|\mathcal{M}| = \left(\prod_{i=1}^n \frac{d}{w_i}\right) \frac{1}{d_{\mathbf{c}}},$$

*where $d_c$ denotes the cardinality of a strong $\mathbf{c} = (1, \ldots, 1)$-class.*

**Proof:** The total number of monomials in

$$\mathcal{W} := \prod_{i=1}^{n} w_i \mathbb{Z}/d\mathbb{Z}$$

is given by the product of the number of possible entries in each position, i.e. $\prod_{i=1}^{n} \frac{d}{w_i}$. Modulo $d$, the weighted degree of an element of $\mathcal{W}$ can take any value in $\{0, \dots, d-1\}$ and the number of elements of $\mathcal{W}$ mapping to the same class of weighted degree modulo $d$ is precisely $\frac{1}{d}|\mathcal{W}|$. Hence, this is the number of elements of weighted degree 0 modulo $d$, i.e.

$$|\mathcal{M}| = \frac{1}{d} \prod_{i=1}^{n} \frac{d}{w_i}.$$

For later considerations, it will be convenient to modify this formula slightly using that $\gcd(w_1, \dots, w_n) = 1$ implies $d = d_c$, which proves the claimed formula.

**q.e.d.**

**Remark 12** *For any given $k \in \{1, \dots, n\}$, we can partition $\mathcal{M}$ into subsets for which the last $(n-k)$ entries coincide. A priori, there are $\prod_{i=k+1}^{n} \frac{d}{w_i}$ possibilities for the last $(n-k)$ entries. As the front part of any element of $\mathcal{M}$, i.e. the first $k$ entries of the element, can only provide weighted degrees which are multiples of $\gcd(w_1, \dots, w_k)$ and as the weighted degree of any element of $\mathcal{M}$ is a multiple of d, not all combinations of the last $(n-k)$ entries can actually occur, but only those which themselves also provide multiples of $\gcd(w_1, \dots, w_k)$ as weighted degree. Hence the total number of these subsets of $\mathcal{M}$ is*

$$\frac{1}{\gcd(w_1, \dots, w_k)} \prod_{i=k+1}^{n} \frac{d}{w_i}.$$

Combining these observations and the lemma, we obtain the following result for the number of strong $\beta$-classes which share the same last $(n-k)$ entries:

**Corollary 13** *Let $\beta \in \mathcal{M}$ satisfy $l(\beta) = 1$ and $\beta_{k+1} = \dots = \beta_n = 0$. Then the number of elements of $\mathcal{M}$ which share the same last $(n-k)$ entries is precisely*

$$\frac{\gcd(w_1, \dots, w_k)}{d} \prod_{i=1}^{k} \frac{d}{w_i}$$

*and the number of strong $\beta$-classes with these last $(n-k)$ entries is*

$$T_\beta = \frac{\gcd(w_1, \dots, w_k)}{d} \frac{\gcd(\beta_1 w_1, \dots, \beta_k w_k)}{d} \prod_{i=1}^{k} \frac{d}{w_i},$$

*which coincides with the total number of singularities in the singular fibre of a 1-parameter family of Fermat-type Calabi–Yau varieties with perturbation term $\underline{x}^\beta$ as considered in section 3.*

Applying this corollary to the two special cases of 1-parameter families considered in 3, we find precisely the number of $A_1$-singularities in the case $\beta = (1, \dots, 1)$ and 1 for the completion of the square. This establishes the first of the two correspondences, which we discuss here. The second one is more subtle and links the Milnor number to the contributions of each $\beta$-class to the zeta-function. It is known, that among the monomials in $\mathcal{M}$ only those that do not contain any entry of the form $\left(\frac{d}{w_i} - 1\right)$ in the $i$-th position should be counted when computing the degree of the associated piece of the zeta-function. Therefore counting the number of possibe ways of constructing such monomials seems a natural question to consider and leads to the following observation:

**Lemma 14** *Let $\beta \in \mathcal{M}$ satisfy $l(\beta) = 1$, $\beta_{k+1} = \cdots = \beta_n = 0$ and $\gcd(w_1, \dots, w_k) = 1$. Then the number of tuples which appear as the last $(n-k)$ entries in an element of $\mathcal{M}$ and do not involve any entry of the form $\left(\frac{d}{w_i} - 1\right)$, is precisely*

$$\prod_{i=k+1}^{n} \left(\frac{d}{w_i} - 1\right).$$

*This coincides with the Milnor number of the appearing singularities according to 3.*

**Proof:** As $\gcd(w_1, \dots, w_k) = 1$, any weighted degree $\sum_{i=k+1}^{n} \alpha_i w_i$ can be completed to a multiple of $d$ by some contribution of the first $k$ entries. Of these only the ones with $\alpha_i \neq \left(\frac{d}{w_i} - 1\right)$ need to be counted which after a direct application of the inclusion-exclusion formula yields the desired expression.

**q.e.d.**

Combining the result of this lemma and the preceding corollary, we see that in the case of $\gcd(w_1, \dots, w_k) = 1$ the total number of strong $\beta$-classes is precisely the total Milnor number. On the other hand, explicit computation showed that for all families of Calabi–Yau 3-folds with one perturbation considered, the degree of the zeta-function drops by exactly the total Milnor number, e.g. for the case of the canonical perturbation, this is the total number of conifold singularities, when passing to a singular fibre. We will see further occurrences of these coincidences in explicit examples for 2-parameter families in the next section.

The correspondence between the findings of the singularity analysis and the intermediate results of the calculation of the zeta-function can be shown to further illuminate the internal structure of the combinatorial objects involved. As the calculations in the general case are rather technical and might block the view for the key observation, we only state this for the case $\beta = (1, \dots, 1)$:

**Remark 15** *By using standard facts about the* gcd*, the cardinality of the set $\mathcal{M}$ can also be stated as*

$$|\mathcal{M}| = \prod_{i=2}^{n} \gcd\left(\frac{d}{w_i}, \frac{d}{\gcd(w_1, \dots, w_{i-1})}\right),$$

*which better reflects the combinatorial structure of* $\mathcal{M}$. [3]. *Consider the first two weights* $w_1$ *and* $w_2$. *The* **c***-subclasses associated to each weight have lengths* $L_1 = \frac{d}{w_1}$, $L_2 = \frac{d}{w_2}$ *respectively. The i-th coordinates of the ordered monomials in every* **c***-class take values in the range* $0, 1, 2, \ldots, \left(\frac{d}{w_i} - 1\right)$ *going up by 1 cyclically. The greatest common divisor of these two* **c***-subclass lengths,* $g_{1,2} = \gcd\left(\frac{d}{w_1}, \frac{d}{w_2}\right)$, *can be used to divide the ranges* $0, 1, 2, \ldots, \left(\frac{d}{w_i} - 1\right)$ *into* $g_{1,2}$ *disjoint partitioning sets given by:*

$$S_{i_k} = \left\{k, k + g_{1,2}, k + 2g_{1,2}, \ldots, k + \left(\frac{L_i}{g_{1,2}} - 1\right) g_{1,2}\right\}, \ 0 \leq k \leq (g_{1,2} - 1).$$

*We can now divide the monomials in* $\mathcal{M}$ *with ith coordinate in* $S_{i_k}$ $(i = 1, 2)$ *into* $g_{1,2}$ *distinct sets. Hence we have established that*

$$g_{1,2} = \gcd\left(\frac{d}{w_1}, \frac{d}{w_2}\right) \mid |\mathcal{M}|,$$

*thus accounting for the first factor in the formula. Iterating this process, we next compute the* **c***-subclass length associated to the pair of weights* $(w_1, w_2)$, *which we shall label* $L_{1,2} = \frac{d}{\gcd(w_1, w_2)}$. *Then we find analogously to the previous step:*

$$g_{(1,2),3} = \gcd\left(\frac{d}{w_3}, \frac{d}{\gcd(w_1, w_2)}\right),$$

*which again leads to a further partitioning. Eventually, this leads to a sequence of refinements of the partitioning which reflects the claimed expression for the number of elements in* $\mathcal{M}$.

# 5 Examples of 2-parameter families

The observations for the 1-parameter families might still be a combinatorial conincidence, but passing to 2-parameter families where the singularity analysis is no longer purely combinatorial we still see the same phenomena: The total Milnor number of a singular fibre matches the change of the degree of the zeta-function when moving from a smooth to a singular fibre. These are precisely the observations which one would expect if Lauder's conjecture of an analogue to the Clemens-Schmid exact sequence holds.

The three considered examples are:

## 5.1 A family in $\mathbb{P}_{(1,1,2,2,2)}$

Considering the family in $\mathbb{P}_{(1,1,2,2,2)}$ given by

$$F = x^8 + y^8 + z^4 + u^4 + v^4 + a \cdot xyzuv + b \cdot x^4 y^4,$$

[3] Note that this decomposition into a product holds for any ordering of the weights.

the discriminant consists of two lines $L_1 = V(b-2)$ and $L_2 = V(b+2)$ (denote $L = L_1 \cup L_2$) and the curve $C$ which possesses the two components $C_1 = V(a^4 - 256b + 512)$ and $C_2 = V(a^4 - 256b - 512)$. For the singular fibres of the family the following singularity types occur:

**$(a,b) \in L \setminus (L \cap C)$:** 4 singularities of type $T_{4,4,4}$ ($\mu = 11$)

**$(a,b) \in C \setminus (C \cap L)$:** 64 ordinary double points

**$(a,b) \in L \cap C$, $a \neq 0$:** 4 singularities of type $T_{4,4,4}$ ($\mu = 11$) and
64 ordinary double points
(transversal intersections of the components of the discriminant)

**$(0,b) \in L \cap C$:** 4 singularities with local normal form $x^2 + z^4 + u^4 + v^4$
($\mu = 27$)
(higher order contact of the components of the discriminant)

When computing the zeta function, we see the following degrees of the contributions depending on the considered fibre of the family. The contributions are labeled by the respective $(1,1,1,1,1)$-classes; classes only differing by a permutation of entries are collected in one line[4]:

| Degree of Contribution $R_{\mathbf{v}}(t)$ According to Singularity | | | | | |
|---|---|---|---|---|---|
| Monomial $\mathbf{v}$ | Perm. | Smooth | 64 $A_1$ with $\mu_{X,x} = 1$ | 4 $T_{4,4,4}$ with $\mu_{X,x} = 11$ | Both or 4 $\mu_{X,x} = 27$ |
| | | $\mu_X = 0$ | $\mu_X = 64$ | $\mu_X = 44$ | $\mu_X = 108$ |
| (0,0,0,0,0) | 1 | 6 | 5 | 4 | 3 |
| (0,2,1,1,1) | 2 | 4 | 3 | 2 | 1 |
| (6,2,0,0,0) | 1 | 4 | 3 | 2 | 1 |
| (0,0,0,2,2) | 3 | 4 | 3 | 3 | 2 |
| (2,0,1,3,3) | 6 | 2 | 1 | 1 | 0 |
| (4,0,2,0,0) | 3 | 4 | 3 | 3 | 2 |
| (0,0,2,1,1) | 3 | 3 | 2 | 2 | 1 |
| (6,0,1,0,0) | 6 | 3 | 2 | 2 | 1 |
| (0,4,0,3,3) | 3 | 4 | 3 | 3 | 2 |
| (4,0,1,1,0) | 3 | 4 | 3 | 3 | 2 |
| (2,0,3,0,0) | 6 | 3 | 2 | 2 | 1 |
| (2,2,1,1,0) | 3 | 3 | 2 | 2 | 1 |
| (0,0,3,1,0) | 6 | 2 | 1 | 2 | 1 |
| (2,0,2,1,0) | 12 | 2 | 1 | 2 | 1 |
| (4,0,2,3,1) | 6 | 0 | -1 | 0 | -1 |
| degree: | | 168 | 104 | 124 | 60 |
| degree change: | | | 64 | 44 | 108 |

The coincidence of the total Milnor number with the total drop in degree as evident in this table, provides experimental evidence for Lauder's conjecture.

[4] We list the number of permutations in the column labeled 'Perm.'

For this first example of an particular family, we also provide the explicit zeta-function in one case, to justify the omission of this data in the later examples. Zeta function data is too richly detailed for the chosen focus of the article. For $p = 7$ and a fibre of the family with 4 $T_{4,4,4}$ singularities, the zeta-function of our family has the following contributions:

| Monomial $\mathbf{v}$ | Contribution | Power $\lambda_{\mathbf{v}}$ |
|---|---|---|
| (0,0,0,0,0) | $(1+18t+2.41pt^2+18p^3t^3+p^6t^4)$ | 1 |
| (0,2,1,1,1) | $(1-pt)(1+pt)$ | 2 |
| (6,2,0,0,0) | $(1-2pt+p^3t^2)$ | 1 |
| (0,0,0,2,2) | $(1+pt)(1+2pt+p^3t^2)$ | 3 |
| (2,0,1,3,3) | $[(1-pt)(1+pt)]^{\frac{1}{2}}$ | 6 |
| (4,0,2,0,0) | $(1+pt)(1+2pt+p^3t^2)$ | 3 |
| (0,0,2,1,1) | $[(1+p^3t^2)(1-pt)(1+pt)]^{\frac{1}{2}}$ | 3 |
| (6,0,1,0,0) | $[(1-2pt+p^3t^2)(1+2pt+p^3t^2)]^{\frac{1}{2}}$ | 6 |
| (0,4,0,3,3) | $[(1+p^3t^2)^2(1-pt)(1+pt)]^{\frac{1}{2}}$ | 3 |
| (4,0,1,1,0) | $[(1+p^3t^2)^2(1-pt)(1+pt)]^{\frac{1}{2}}$ | 3 |
| (2,0,3,0,0) | $[(1-2pt+p^3t^2)(1+2pt+p^3t^2)]^{\frac{1}{2}}$ | 6 |
| (2,2,1,1,0) | $[(1+p^3t^2)(1-pt)(1+pt)]^{\frac{1}{2}}$ | 3 |
| (0,0,3,1,0) | $(1-pt)(1+pt)$ | 6 |
| (2,0,2,1,0) | $[(1-2pt+p^3t^2)(1+2pt+p^3t^2)]^{\frac{1}{2}}$ | 12 |
| (4,0,2,3,1) | 1 | 6 |

Note that the second roots arise from the algorithmic computation of the zeta function, but never occur in the final result, because the corresponding contributions always arise in pairs.

### 5.2 A family in $\mathbb{P}_{(1,1,2,2,6)}$

Considering the family in $\mathbb{P}_{(1,1,2,2,6)}$ given by

$$F = x^{12} + y^{12} + z^6 + u^6 + v^2 + a \cdot xyzuv + b \cdot x^6 y^6,$$

the discriminant consists of two lines $L_1 = V(b-2)$ and $L_2 = V(b+2)$ (denote $L = L_1 \cup L_2$) and the curve $C$ which possesses the two components $C_1 = V(a^6 - 1728b + 3456)$ and $C_2 = V(a^6 - 1728b - 3456)$. For the singular fibres of the family the following singularity types occur:

**$(a,b) \in L \setminus (L \cap C)$:** 6 singularities of type $T_{2,6,6} = Y^1_{2,2}$ ($\mu = 13$)

**$(a,b) \in C \setminus (C \cap L)$:** 72 ordinary double points

**$(a,b) \in L \cap C$, $a \neq 0$:** 6 singularities of type $T_{2,6,6}$ ($\mu = 13$) and 72 ordinary double points (transversal intersections of the components of the discriminant)

**$(0,b) \in L \cap C$:** 6 singularities with local normal form $x^2 + z^6 + u^6 + v^2$ ($\mu = 25$) (higher order contact of the components of the discriminant)

Here the contributions to the factors of the zeta-function are the following:

| Degree of Contribution $R_{\mathbf{v}}(t)$ According to Singularity | | | | | |
|---|---|---|---|---|---|
| Monomial $\mathbf{v}$ | Perm. | Smooth | 72 $A_1$ with $\mu_{X,x} = 1$ | 6 $T_{2,6,6}$ with $\mu_{X,x} = 13$ | Both or 6 $\mu_{X,x} = 25$ |
| | | $\mu_X = 0$ | $\mu_X = 72$ | $\mu_X = 78$ | $\mu_X = 150$ |
| (0,0,0,0,0) | 1 | 6 | 5 | 4 | 3 |
| (11,1,0,0,0) | 2 | 4 | 3 | 2 | 1 |
| (10,2,0,0,0) | 2 | 6 | 5 | 4 | 3 |
| (9,3,0,0,0) | 1 | 4 | 3 | 2 | 1 |
| (10,0,0,1,0) | 4 | 4 | 3 | 3 | 2 |
| (9,1,0,1,0) | 4 | 3 | 2 | 2 | 1 |
| (8,2,0,1,0) | 2 | 4 | 3 | 3 | 2 |
| (5,5,0,1,0) | 2 | 3 | 2 | 2 | 1 |
| (8,0,2,0,0) | 4 | 6 | 5 | 4 | 3 |
| (7,1,2,0,0) | 2 | 4 | 3 | 2 | 1 |
| (5,3,2,0,0) | 4 | 4 | 3 | 2 | 1 |
| (4,4,2,0,0) | 2 | 6 | 5 | 4 | 3 |
| (6,0,3,0,0) | 2 | 4 | 3 | 3 | 2 |
| (5,1,3,0,0) | 4 | 2 | 1 | 1 | 0 |
| (4,2,3,0,0) | 4 | 4 | 3 | 3 | 2 |
| (3,3,3,0,0) | 2 | 4 | 3 | 3 | 2 |
| (6,0,0,0,1) | 1 | 4 | 3 | 3 | 2 |
| (5,1,0,0,1) | 2 | 4 | 3 | 3 | 2 |
| (4,2,0,0,1) | 2 | 2 | 1 | 1 | 0 |
| (3,3,0,0,1) | 1 | 4 | 3 | 3 | 2 |
| (2,2,0,1,1) | 2 | 3 | 2 | 2 | 1 |
| (3,1,0,1,1) | 4 | 4 | 3 | 3 | 2 |
| (10,6,0,1,1) | 4 | 3 | 2 | 2 | 1 |
| (11,5,0,1,1) | 2 | 4 | 3 | 3 | 2 |
| (1,1,2,0,1) | 2 | 2 | 1 | 2 | 1 |
| (2,0,2,0,1) | 4 | 2 | 1 | 2 | 1 |
| (9,5,2,0,1) | 4 | 2 | 1 | 2 | 1 |
| (10,4,2,0,1) | 2 | 0 | -1 | 0 | -1 |
| degree: | | 254 | 182 | 176 | 104 |
| degree change: | | | 72 | 78 | 150 |

## 5.3 A family in $\mathbb{P}_{(1,1,3,3,4)}$

Considering the family in $\mathbb{P}_{(1,1,3,3,4)}$ given by

$$F = x^{12} + y^{12} + z^4 + u^4 + v^3 + a \cdot xyzuv + b \cdot x^4 y^4 v,$$

the discriminant consists of three lines $L = V(b^3 + 27)$ and the curve $C = V(a^{12} - a^8 b^4 - 576 a^8 b + 512 a^4 b^5 + 96768 a^4 b^2 - 65536 b^6 - 3538944 b^3 - 47775744)$. For the singular fibres of the family the following singularity types occur:

$(a,b) \in L \setminus (L \cap C)$**:** 12 ordinary double points

$(a,b) \in C \setminus ((L \cap C) \cup C_{sing})$**:** 48 ordinary double points

$(0,b) \in L \cap C$**:** 12 singularities of type $X_9$
(higher order contact of components of the dicriminant)

$(a,b) \in L \cap C, a \neq 0$**:** 60 ordinary double points
(transversal intersections of the components of the discriminant)

$(a,b) \in V(9a^4 - 16b^4, b^3 - 108) \subset C_{sing}$**:** 48 $A_2$ singularities

$(a,b) \in V(a^4 - 288b, b^3 - 216) \subset C_{sing}$**:** 96 $A_1$ singularities

| Degree of Contribution $R_{\mathbf{v}}(t)$ According to Singularity | | | | | | | |
|---|---|---|---|---|---|---|---|
| Monomial $\mathbf{v}$ | Perm. | Smooth | 12 $A_1$ | 48 $A_1$ | 60 $A_1$ | 48 $A_2$ | 12 $X_9$ |
| (0,0,0,0,0) | 1 | 6 | 5 | 5 | 4 | 4 | 3 |
| (11,1,0,0,0) | 2 | 4 | 3 | 3 | 2 | 2 | 1 |
| (10,2,0,0,0) | 2 | 4 | 3 | 3 | 2 | 2 | 1 |
| (9,3,0,0,0) | 2 | 6 | 5 | 5 | 4 | 4 | 3 |
| (8,4,0,0,0) | 2 | 6 | 5 | 5 | 4 | 4 | 3 |
| (7,5,0,0,0) | 2 | 4 | 3 | 3 | 2 | 2 | 1 |
| (6,6,0,0,0) | 1 | 6 | 5 | 5 | 4 | 4 | 3 |
| (9,0,1,0,0) | 4 | 4 | 4 | 3 | 3 | 2 | 2 |
| (8,1,1,0,0) | 4 | 4 | 4 | 3 | 3 | 2 | 2 |
| (7,2,1,0,0) | 4 | 2 | 2 | 1 | 1 | 0 | 0 |
| (6,3,1,0,0) | 4 | 4 | 4 | 3 | 3 | 2 | 2 |
| (5,4,1,0,0) | 4 | 4 | 4 | 3 | 3 | 2 | 2 |
| (11,10,1,0,0) | 4 | 2 | 2 | 1 | 1 | 0 | 0 |
| (6,0,2,0,0) | 2 | 4 | 4 | 3 | 3 | 2 | 2 |
| (5,1,2,0,0) | 4 | 2 | 2 | 1 | 1 | 0 | 0 |
| (4,2,2,0,0) | 4 | 4 | 4 | 3 | 3 | 2 | 2 |
| (3,3,2,0,0) | 2 | 4 | 4 | 3 | 3 | 2 | 2 |
| degree: | | 180 | 168 | 132 | 120 | 84 | 72 |
| degree change: | | | 12 | 48 | 60 | 96 | 108 |

# 6 Conclusion

For one-parameter families it has been shown that the combinatorics of the monomial equivalence classes, which split up the zeta function, is intimately related to the singularity structure of the varieties. Moreover, all computed examples[5] have also shown that the change of degree of the

[5]In addition to the examples stated in this article, all Calabi–Yau 3-folds of Fermat-type have been systematically studied combinatorially from our point of view. For a number of interesting cases, which did not pose too many computational difficulties for the Mathematica programs, the explicit zeta-functions have been determined for low primes – all showing the same behaviour. We choose to include only 3 explicit examples of 2-parameter families which already cover most of our observations, because adding further examples would not show new phenomena.

contribution by each labeled part of the zeta function follows patterns of the set of strong $\beta$-classes. The total change of degree of the zeta function upon passing to a singular fibre has been observed as coinciding with the total Milnor number of the singular fibre.

For the more involved case of two-parameter families, it is also apparent from the finite number of cases computed, that the combinatorics of the strong equivalence classes once again seem to be reflected in the singularity structure. From this arises the following conjecture, which strongly refines the conjectures of [K04, K06]:

**Conjecture 1 (Singularity -geometric/combinatoric duality)** *Given a family of Calabi–Yau varieties with special fibre of Fermat type, the total Milnor number of each arising singular fibre is expressible in terms of the change of the degree of the zeta function when passing to the singular fibre.*
*The singularity structure as reflected in the relative Milnor (and Tjurina) algebra of the family encodes information on the degree changes of factors of the zeta function labeled by $\beta$-classes.*

The degenerative properties of the zeta functions at singular points studied here (and the global L-series they give rise to) were recently exploited in [KLS] in order to investigate the phenomenon of 'string modularity'. The main result was that for several families (all containing a Fermat member as a special fibre), the modular form associated to part of the global zeta function or L-series found at a degenerate, non-Fermat point in the moduli space agreed with that of the motivic L-series of a different weighted Fermat variety. These pairs are called L-correlated and provide evidence that the conformal field theory at deformed fibres (currently difficult to define) are related to those of the well-defined rational conformal field theories of Fermat-type manifolds (Gepner models) with a completely different geometry. Our singularity-theoretic and combinatorial results would the aid exploration of both finding more examples of singular members of Calabi–Yau families exhibiting modularity, and perhaps more L-correlated 'string-modular' pairs.

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